

# Noncompact $sl(N)$ spin chains: BGG – resolution, $Q$ -operators and alternating sum representation for finite dimensional transfer matrices.

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## Abstract

We study properties of transfer matrices in the  $sl(N)$  spin chain models. The transfer matrices with an infinite dimensional auxiliary space are factorized into the product of  $N$  commuting Baxter  $Q$ -operators. We consider the transfer matrices with auxiliary spaces of a special type (including the finite dimensional ones). It is shown that they can be represented as the alternating sum over the transfer matrices with infinite dimensional auxiliary spaces. We show that certain combinations of the Baxter  $Q$ -operators can be identified with the  $Q$ -functions which appear in the Nested Bethe Ansatz.

# 1 Introduction

The  $\mathcal{R}$ -matrix approach [1–4] to the theory of integrable spin chain models allows one to construct an infinite set of commuting operators – transfer matrices which are defined as the trace of monodromy matrix over an auxiliary space which can be both finite and infinite dimensional one. Transfer matrices obey certain functional relations — the so - called fusion relations [2, 5, 6] which provide one with a powerful method to analyze transfer matrices with finite dimensional auxiliary space (= finite dimensional transfer matrices) (see e.g. Refs. [7–11]). The derivation of fusion relations is based heavily on the properties of tensor products of representations. Namely, in much the same way as any finite - dimensional representation (we restrict ourselves to the case of  $sl(N)$  algebra) can be obtained by tensoring the fundamental representations any finite - dimensional transfer matrix can be expressed in terms of the transfer matrices with special representations in the auxiliary space [5, 12, 13].

The alternative approach is to start the analysis with consideration of transfer matrices with infinite dimensional auxiliary spaces of a general type, which we will refer to as the generic transfer matrices. All other transfer matrices, including finite-dimensional ones, can be obtained from the generic transfer matrices by some reduction procedure. The main advantage of this approach is a surprisingly simple structure of the generic transfer matrices. Namely, it was shown in Ref. [14] that the generic  $sl(N)$  transfer matrix factorizes into the product of  $N$  commuting operators \*

$$\mathsf{T}_\rho(u) \sim \mathcal{Q}_1(u + \rho_1) \mathcal{Q}_2(u + \rho_2) \dots \mathcal{Q}_N(u + \rho_N). \quad (1.1)$$

The parameters  $\{\rho_k\}$  define the representation of the  $sl(N)$  algebra in the auxiliary space (all notations will be discussed in sect. 2). The operators  $\mathcal{Q}_k(u)$  form a commutative family and usually referred to as the Baxter  $\mathcal{Q}$ -operators. All dependence of the generic transfer matrix on the representation in the auxiliary space comes through the shifts of the spectral parameters of the factorizing (Baxter) operators. Thus, Eq. (1.1) gives a complete description of generic transfer matrices in terms of  $N$  Baxter  $\mathcal{Q}$ -operators.

Representations of  $sl(N)$  algebra are not exhausted by the finite dimensional and generic infinite dimensional representations, however. For special values of the parameters  $\{\rho_k\}$  the generic representation becomes reducible and contains invariant subspaces, which can be both finite and infinite dimensional. In particular any finite dimensional representation is realized as an invariant subspace of some generic representation. In this work we study transfer matrices with an auxiliary space of a special type and show that they can be expressed in terms of the generic transfer matrices. In particular we obtain the determinant formula for the finite - dimensional transfer matrices  $t_\rho(u)$

$$t_\rho(u) \sim \det \begin{vmatrix} \mathcal{Q}_1(u + \rho_1) & \mathcal{Q}_1(u + \rho_2) & \dots & \mathcal{Q}_1(u + \rho_N) \\ \mathcal{Q}_2(u + \rho_1) & \mathcal{Q}_2(u + \rho_2) & \dots & \mathcal{Q}_2(u + \rho_N) \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{Q}_N(u + \rho_1) & \mathcal{Q}_N(u + \rho_2) & \dots & \mathcal{Q}_N(u + \rho_N) \end{vmatrix}. \quad (1.2)$$

This formula is a direct consequence of the factorized representation (1.1) and the Bernstein - Gelfand - Gelfand resolution for the finite dimensional modules [17]. The determinant formula (1.2) is quite natural from many points of view and, to our knowledge, was first proposed

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\*This property is a direct consequence of the factorization property of the  $sl(N)$  invariant  $\mathcal{R}$ -operator [14–16].

in Ref. [18,19]. It contains in a concise form a lot of information about the spin chain model. In particular, it can be shown that the formula (1.2) implies the Nested Bethe Ansatz equations for the eigenstates of transfer matrices.

Equations (1.1) and (1.2) show that both generic and finite dimensional transfer matrices can be expressed in terms of Baxter  $\mathcal{Q}$ -operators. The Baxter operators were object of an intensive study in the last decade. The method of Baxter operators allows to construct solutions for models which do not possess the pseudovacuum state and cannot be solved by the Algebraic Bethe Ansatz (ABA). The first construction of such operator was given in the seminal paper of Baxter [20]. Later on nontrivial examples of Baxter operators were found for a number of models. These are, mostly, models with a rank one symmetry algebra. Common approach for constructing Baxter operators is based, in this case, on solving the so-called  $T - Q$  relation [21–35]. Straightforward attempts to apply such approach for the models with higher rank symmetry groups encounter certain difficulties. Another approach, the so-called  $q$ -oscillators approach [18, 36–41], provides a regular method for constructing Baxter  $\mathcal{Q}$ -operators. In certain aspects it is close to the approach based on the  $\mathcal{R}$ -matrix factorization [14–16, 42–44].

The paper is organized as follows: In section 2 we explain notations and discuss properties of  $sl(N)$  modules and intertwining operators. In section 3 we remind the definition of  $sl(N)$  transfer matrices and Baxter  $\mathcal{Q}$ -operators and formulate our results for the transfer matrices with auxiliary spaces of a special type. In section 4 we derive the alternating sum representation (determinant formula) for the transfer matrices of this type. In section 5 the relation of the constructed transfer matrices with Nested Bethe Ansatz is discussed. Some technical details are collected in Appendix.

## 2 $sl(N)$ modules

Let  $\mathbb{V}$  be a vector space of polynomials of  $N(N-1)/2$  complex variables,  $z_{ki}$ ,  $1 \leq i < k \leq N$  of arbitrary degree

$$\mathbb{V} = \left\{ p(z_{21}, z_{31}, \dots, z_{NN-1}), \quad \deg(p) < \infty \right\}. \quad (2.1)$$

We define the lower triangular matrix

$$z = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ z_{21} & 1 & 0 & \dots & 0 \\ z_{31} & z_{32} & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z_{N1} & z_{N2} & \dots & z_{NN-1} & 1 \end{pmatrix} \quad (2.2)$$

and adopt a shorthand notation  $p(z) \equiv p(z_{21}, z_{31}, \dots, z_{NN-1})$ .

Let  $D_{mn}$ ,  $m > n$  and  $E_{ik}$  be the following differential operators on  $\mathbb{V}$ :

$$\begin{aligned} D_{mn} &= \sum_{j=m}^N z_{jm} \frac{\partial}{\partial z_{jn}}, \\ E_{ik} &= - \sum_{m \leq n} z_{km} \left( D_{nm} + \delta_{nm} \rho_m \right) (z^{-1})_{ni}, \end{aligned} \quad (2.3)$$

where  $\rho_k$ ,  $k = 1, \dots, N$  are complex numbers subjected to the constraint

$$\rho_1 + \rho_2 + \dots + \rho_N = N(N-1)/2. \quad (2.4)$$

Let  $e_{ki}$  be the generators of the  $sl(N)$  algebra,

$$[e_{ik}, e_{mn}] = \delta_{km} e_{in} - \delta_{in} e_{mk}.$$

The homomorphism  $\pi^\rho$

$$\pi^\rho : e_{ik} \rightarrow E_{ik}, \quad (2.5)$$

where  $\rho$  is the  $N$  dimensional vector,  $\rho \equiv (\rho_1, \dots, \rho_N)$ , defines a representation of the  $sl(N)$  algebra on the space  $\mathbb{V}$  or provides  $\mathbb{V}$  with a structure of the  $sl(N)$  module. We will denote such a module by  $\mathbb{V}_\rho$ .

## 2.1 Submodules and intertwining operators

The representation  $\pi^\rho$  (the module  $\mathbb{V}_\rho$ ) is *irreducible* if none of the differences  $\rho_{ik} = \rho_i - \rho_k$ ,  $i < k$ , is a positive integer [17, 45]. Otherwise, the module  $\mathbb{V}_\rho$  has invariant subspaces. Some of them can be obtained as kernels of intertwining operators.

The operator  $D_{k+1k}$  raised to the power  $\rho_{kk+1}$  intertwines the generators corresponding to two different sets of the parameters  $\rho$ , namely

$$D_{k+1k}^{\rho_{kk+1}} E_{nm}(\rho) = E_{nm}(\rho') D_{k+1k}^{\rho_{kk+1}}, \quad (2.6)$$

where  $\rho'$  differs from  $\rho$  by permutation of  $\rho_k$  and  $\rho_{k+1}$ :  $\rho' = (\dots, \rho_{k+1}, \rho_k, \dots) = P_{kk+1}\rho$ . Whenever  $\rho_{kk+1} \in \mathbb{N}$ , the operator  $D_{k+1k}^{\rho_{kk+1}}$  is a well defined operator on  $\mathbb{V}$ . It intertwines the representations  $\pi^\rho$  and  $\pi^{\rho'}$  and its kernel is an invariant subspace of  $\mathbb{V}$ .

We adopt a shorthand notation for the operators  $D_{k+1,k}$ ,  $D_k \equiv D_{k+1,k}$ ,  $k = 1, \dots, N-1$ . The operators commute,  $D_k D_i = D_i D_k$  if  $|i-k| > 1$ , while adjacent operators satisfy the following relations

$$D_k^a D_{k+1}^{a+b} D_k^b = D_{k+1}^b D_k^{a+b} D_{k+1}^a. \quad (2.7)$$

These operators serve as elementary building blocks for constructing more general intertwining operators. Let  $\rho_{ik}$  be a vector obtained from  $\rho$  by a permutation of  $i$ -th and  $k$ -th elements,  $\rho_{ik} = P_{ik}\rho$ . The operator  $U_{ik}$  which intertwines the generators  $E(\rho)$  and  $E(\rho_{ik})$  has the form (we assume that  $i < k$ )

$$U_{ik}(\rho) = (D_{k-1}^{\rho_{ik-1}} \dots D_{i+1}^{\rho_{ii+1}}) D_i^{\rho_{ik}} (D_{i+1}^{\rho_{i+1k}} \dots D_{k-1}^{\rho_{k-1k}}). \quad (2.8)$$

The statement that  $U_{ik}$  is an intertwining operator follows from (2.6). It is less obvious that this operator is a *polynomial* in  $D_{nm}$  if  $\rho_{ik} = n > 0$  and therefore can be viewed as an operator on  $\mathbb{V}$ . This can be easily checked with the help of commutation relations for the operators  $D_{mn}$

$$[D_{mn}, D_{ik}] = \delta_{in} D_{mk} - \delta_{mk} D_{in}. \quad (2.9)$$

Thus, whenever the difference  $\rho_i - \rho_k$ ,  $i < k$  is a positive integer the operator  $U_{ik}$  intertwines the representations  $\pi^\rho$  and  $\pi^{\rho_{ik}}$ ,  $\pi^{\rho_{ik}} U_{ik} = U_{ik} \pi^\rho$ .

In a general situation (see Ref. [17], Theorem 8.8), the operator  $U$  which intertwines the generators in the representations  $\pi^\rho$  and  $\pi^{\rho'}$ , is a well defined operator on  $\mathbb{V}$  if and only if

- $\rho'$  can be represented in the following form

$$\rho' = P_{i_n j_n} \dots P_{i_1 j_1} \rho, \quad (i_k < j_k)$$

- All differences  $\rho_{i_{k+1}}^{(k)} - \rho_{j_{k+1}}^{(k)} = m_{k+1} \in \mathbb{N}$ . Here  $\rho^{(k)} = P_{i_k j_k} \dots P_{i_1 j_1} \rho$ .

Evidently, the kernel of an intertwining operator is an invariant subspace of  $\mathbb{V}$ . We will be interested in invariant subspaces of a special type. Namely, let us consider a situation when the differences  $\rho_{mm+1}$  are natural numbers starting from  $m = N - k$ :

$$\rho_{mm+1} = n_m \in \mathbb{N}, \quad \text{for} \quad m = N - k, \dots, N - 1. \quad (2.10)$$

In this case all operators  $D_m^{\rho_{mm+1}}$  with  $m \geq N - k$  have nontrivial kernels. Let us define an invariant submodule  $\mathbb{V}_\rho^{(k)}$  as their intersection

$$\mathbb{V}_\rho^{(k)} = \ker D_{N-k}^{\rho_{N-k} N-k+1} \cap \dots \cap \ker D_{N-2}^{\rho_{N-2} N-1} \cap \ker D_{N-1}^{\rho_{N-1} N}. \quad (2.11)$$

The index  $k$  shows the number of the intersecting spaces in (2.11). We will refer to  $\mathbb{V}_\rho^{(k)}$  as the space of the  $k$ -th level. Thus, a zero level space is the generic space itself,  $\mathbb{V}_\rho^{(0)} \equiv \mathbb{V}_\rho$ , a space of the first level  $\mathbb{V}_\rho^{(1)} = \ker D_{N-1}^{\rho_{N-1} N}$  and so on. The higher level of the space is the more restrictions the functions from this space satisfy. The space of the highest level for a given  $N$ ,  $\mathbb{V}_\rho^{(N-1)}$ , is an irreducible finite dimensional  $sl(N)$  module (see Ref. [46], Chapter X).

Taking into account that operator  $D_k$

$$D_k = \sum_{j=k+1}^N z_{jk+1} \frac{\partial}{\partial z_{jk}} \quad (2.12)$$

depends only on the elements in the  $k, k+1$ -th columns of the matrix  $z$  one finds that for  $k < N - 1$  the subspace  $\mathbb{V}_\rho^{(k)}$  can be represented as a tensor product of an infinite dimensional and finite dimensional spaces,

$$\mathbb{V}_\rho^{(k)} = \mathbb{V}_k \otimes v_k. \quad (2.13)$$

The elements of the space  $\mathbb{V}_k$  are polynomials of arbitrary degree which depend on the variables in the first  $N - k - 1$ -columns of the matrix  $z$ . Elements of the finite dimensional space  $v_k$  are polynomials which depend on the variables in the last  $k + 1$  columns of the matrix  $z$  and are nullified by the operators  $D_m^{\rho_{mm+1}}$ ,  $m \geq N - k$ . The space  $v_k$  can be considered as a finite - dimensional  $sl(k + 1)$  module  $v_k = v_{\rho_{N-k} \dots \rho_N}$ . We will be interested in the transfer matrices with the higher level space  $\mathbb{V}_\rho^{(k)}$  as an auxiliary space.

### 3 Transfer matrices

By a definition the  $\mathcal{R}$ -operator is a solution of the Yang-Baxter equation (YBE)

$$\mathcal{R}_{12}(u) \mathcal{R}_{13}(u + v) \mathcal{R}_{23}(v) = \mathcal{R}_{23}(v) \mathcal{R}_{13}(u + v) \mathcal{R}_{12}(u). \quad (3.1)$$

It was shown in Ref. [14] that the  $sl(N)$  invariant  $\mathcal{R}$ -operator acting on the tensor product of two generic  $sl(N)$  modules  $\mathbb{V}_{\boldsymbol{\sigma}} \otimes \mathbb{V}_{\boldsymbol{\rho}}$  ( $\boldsymbol{\sigma} = \{\sigma_1, \dots, \sigma_N\}$ ,  $\boldsymbol{\rho} = \{\rho_1, \dots, \rho_N\}$ )

$$\begin{aligned} \mathcal{R}_{12}(u) : \mathbb{V}_{\boldsymbol{\sigma}} \otimes \mathbb{V}_{\boldsymbol{\rho}} &\mapsto \mathbb{V}_{\boldsymbol{\sigma}} \otimes \mathbb{V}_{\boldsymbol{\rho}}, \\ [E_{ik}^{(\boldsymbol{\sigma})} + E_{ik}^{(\boldsymbol{\rho})}, \mathcal{R}_{12}(u)] &= 0 \end{aligned} \quad (3.2)$$

can be represented in the factorized form [14, 16]

$$\mathcal{R}_{12}(u) = P_{12} \mathbb{R}_{12}^{(1)}(u - \sigma_1 + \rho_1) \mathbb{R}_{12}^{(2)}(u - \sigma_2 + \rho_2) \dots \mathbb{R}_{12}^{(N)}(u - \sigma_N + \rho_N). \quad (3.3)$$

Here  $P_{12}$  is the permutation operator and  $\mathbb{R}_{12}^{(k)}(u)$  are the factorizing operators.

The transfer matrix is defined as the trace of the monodromy matrix over the auxiliary space. In the case of an infinite dimensional auxiliary space one has to ensure the convergence of the trace. To this end we consider the modified  $\mathcal{R}$  operator [14]

$$\mathcal{R}_{12}(u, \tau) \equiv \mathcal{R}_{12}(u, \tau_1, \dots, \tau_{N-1}) = \tau^H \mathcal{R}_{12}(u), \quad (3.4)$$

where  $\tau^H$  is a shorthand notation for the following operator acting on the second space in the tensor product  $\mathbb{V}_{\boldsymbol{\sigma}} \otimes \mathbb{V}_{\boldsymbol{\rho}}$

$$\tau^H \equiv \tau_1^{H_1} \tau_2^{H_2} \dots \tau_{N-1}^{H_{N-1}} = \prod_{p=1}^{N-1} \tau_p^{H_p}. \quad (3.5)$$

The operators  $H_p$  are defined as follows <sup>†</sup>

$$H_p = \sum_{k=1}^p (E_{kk} + k - N) = \sum_{k=1}^p \left( -\rho_k + \sum_{m=p+1}^N z_{mk} \partial_{z_{mk}} \right). \quad (3.6)$$

The operators  $\mathcal{R}_{12}(u, \tau)$  obey YBE

$$\mathcal{R}_{12}(u, \tau) \mathcal{R}_{13}(v, \tau) \mathcal{R}_{23}(v - u) = \mathcal{R}_{23}(v - u) \mathcal{R}_{13}(v, \tau) \mathcal{R}_{12}(u, \tau) \quad (3.7)$$

and serve to construct the transfer matrix

$$\mathbb{T}_{\boldsymbol{\rho}}(u, \tau) = \text{tr}_{\boldsymbol{\rho}} \left\{ \mathcal{R}_{10}(u, \tau) \dots \mathcal{R}_{L0}(u, \tau) \right\}. \quad (3.8)$$

Here the index  $\boldsymbol{\rho}$  specifies the representation,  $\pi^{(0)} = \pi^{\boldsymbol{\rho}}$ , of the  $sl(N)$  algebra on the auxiliary space. The trace on the rhs of Eq. (3.8) exists for  $\tau < 1$  ( $\tau_k < 1, k = 1, \dots, N-1$ ) and gives rise to a well-defined operator  $\mathbb{T}_{\boldsymbol{\rho}}(u, \tau)$ , (for details see Ref. [14]). For the sake of simplicity we will consider homogeneous spin chains only, i.e. assume that the representations of the  $sl(N)$  algebra on the quantum space at each site are equivalent,  $\pi^{\boldsymbol{\sigma}^1} = \pi^{\boldsymbol{\sigma}^2} = \dots = \pi^{\boldsymbol{\sigma}^L} \equiv \pi^{\boldsymbol{\sigma}}$ . In this case the transfer matrix (3.8) is factorized into the product of the Baxter  $\mathcal{Q}$ -operators

$$\mathbb{T}_{\boldsymbol{\rho}}(u, \tau) = (\mathcal{P} \tau^H)^{N-1} \mathcal{Q}_1(u + \rho_1, \tau) \mathcal{Q}_2(u + \rho_2, \tau) \dots \mathcal{Q}_N(u + \rho_N, \tau), \quad (3.9)$$

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<sup>†</sup>The definition of the operators  $H_k$  adopted here is differ from that in [14] by a constant. It is done to have simple commutation relations between the  $\mathcal{R}$  matrix and intertwining operators (see Eq. (3.19))

where the operator  $H$  ( $H = (H_1, \dots, H_{N-1})$ ) acts on the quantum space of the model  $\mathbb{V}_q = \mathbb{V}_{\sigma} \otimes \dots \otimes \mathbb{V}_{\sigma}$ ,

$$H_k = H_k^{(1)} + \dots + H_k^{(L)} \quad (3.10)$$

and  $\mathcal{P}$  is the operator of cyclic permutations

$$\mathcal{P} P(z_1, \dots, z_L) = P(z_L, z_1, \dots, z_{L-1}). \quad (3.11)$$

The definition of the operators  $\mathcal{Q}_k(u, \tau)$  mimics the definition of transfer matrix

$$\mathcal{Q}_k(u + \sigma_k, \tau) = \text{tr}_0 \left\{ R_{10}^{(k)}(u, \tau) \dots R_{L0}^{(k)}(u, \tau) \right\}. \quad (3.12)$$

The operators  $R_{i0}^{(k)}(u, \tau)$  are expressed in terms of the factorizing operators (3.3) as follows

$$R_{i0}^{(k)}(u, \tau) = \tau^H P_{i0} \mathbb{R}_{i0}^{(k)}(u), \quad (3.13)$$

where the operator  $H$  (see Eq. (3.6)) acts on the auxiliary space. The Baxter  $\mathcal{Q}$ -operators commute with each other and also with the operator of cyclic permutation  $\mathcal{P}$  and the operators  $H_k$  (3.10)

$$[\mathcal{Q}_k(u, \tau), \mathcal{Q}_m(v, \tau)] = [\mathcal{Q}_k(u, \tau), \mathcal{P}] = [\mathcal{Q}_k(u, \tau), H_k] = 0. \quad (3.14)$$

They also satisfy the simple normalization condition

$$\mathcal{Q}_k(\sigma_k, \tau) = \mathcal{P} \tau^H, \quad (3.15)$$

where  $\sigma_k$  are the parameters of the representation in the quantum space,  $\sigma = (\sigma_1, \dots, \sigma_N)$ .

Note that the factorized representation (3.9) for the transfer matrix holds for arbitrary representation  $\pi^{\rho}$  in the auxiliary space, whether it is irreducible or not. In case that the representation  $\pi^{\rho}$  is reducible the restriction of the  $\mathcal{R}$  operator to the invariant subspace  $\mathbb{V}'$  gives rise to a new solution of the YBE,  $\mathcal{R}'$ :

$$\mathcal{R}' : \mathbb{V} \otimes \mathbb{V}' \mapsto \mathbb{V} \otimes \mathbb{V}'.$$

In this way one can obtain all  $sl(N)$  invariant solutions of the YBE starting from the generic  $\mathcal{R}$  operator. For a reducible module  $\mathbb{V}_{\rho}$  the  $\mathcal{R}$ -operator has the block triangular form,

$$\mathcal{R} = \begin{pmatrix} \mathcal{R}' & \star \\ 0 & \mathcal{R}'' \end{pmatrix}, \quad (3.16)$$

where  $\mathcal{R}''$  is an  $\mathcal{R}$ -operator on  $\mathbb{V} \otimes \mathbb{V}''$  and  $\mathbb{V}''$  is the factor space  $\mathbb{V}'' = \mathbb{V}_{\rho}/\mathbb{V}'$ . Thus the trace over  $\mathbb{V}_{\rho}$  in Eq. (3.8) decays into the traces over  $\mathbb{V}'$  and  $\mathbb{V}''$ , so that one gets

$$\mathbb{T}_{\rho}(u, \tau) = \mathbb{T}'_{\rho}(u, \tau) + \mathbb{T}''_{\rho}(u, \tau). \quad (3.17)$$

Studying reducible representations one can, in principle, express transfer matrices with arbitrary  $sl(N)$  submodule as the auxiliary space in terms of the generic transfer matrices,  $\mathbb{T}_{\rho}(u, \tau)$ . In a general situation the solution of the problem is not known, however for the transfer matrices of a special type which will be considered in the next subsection such expression exists.

### 3.1 Higher level transfer matrices

Let consider an  $\mathcal{R}$ -operator on the tensor product  $\mathbb{V}_\sigma \otimes \mathbb{V}_\rho$  with  $\rho$  satisfying conditions (2.10). The generic module  $\mathbb{V}_\rho$  has an invariant submodule  $\mathbb{V}_\rho^{(k)}$ . Let the operator  $\mathcal{R}^{(k)}$  be a restriction of the  $\mathcal{R}$ -operator to the subspace  $\mathbb{V}_\sigma \otimes \mathbb{V}_\rho^{(k)}$ . We define a transfer matrix of  $k$ -th level,  $\mathsf{T}_\rho^{(k)}$ , as follows <sup>‡</sup>

$$\mathsf{T}_\rho^{(k)}(u, \tau) = \text{tr}_{\mathbb{V}_\rho^{(k)}} \mathcal{R}_{10}^{(k)}(u, \tau) \dots \mathcal{R}_{L0}^{(k)}(u, \tau). \quad (3.18)$$

We also will use a special notation for finite dimensional transfer matrices,  $t_\rho(u, \tau) = \mathsf{T}_\rho^{(N-1)}(u, \tau)$ . For later convenience we will also assume that the normalization of the  $\mathcal{R}$ -operator is chosen in such a way that it has simple commutation relations with the intertwining operators <sup>§</sup>

$$U \mathcal{R}_{\sigma\rho}(u) = \mathcal{R}_{\sigma\rho'}(u) U, \quad (3.19)$$

where  $U$  intertwines the representations  $\pi^\rho$  and  $\pi^{\rho'}$ . Existence of such normalization is provided by the universal  $\mathcal{R}$ -operator construction [47]. Indeed, the universal  $\mathcal{R}$ -operator is a function of algebra generators only and hence satisfies (3.19).

The higher level transfer matrices have the following properties:

- The transfer matrix (3.18) of the  $k$ -th level can be represented in the factorized form

$$\mathsf{T}_\rho^{(k)}(u, \tau) = (\mathcal{P}\tau^H)^{k+1-N} \prod_{j=1}^{N-k-1} \mathcal{Q}_j(u + \rho_j, \tau) W_{\rho_k}^{(k)}(u, \tau), \quad (3.20)$$

where  $\rho_k = (\rho_{N-k}, \dots, \rho_N)$ . The operator  $W_{\rho_k}^{(k)}(u, \tau)$  is given by the trace of a special monodromy matrix over the auxiliary space  $\mathbb{V}_\rho^{(k)}$

$$W_{\rho_k}^{(k)}(u, \tau) = \text{tr}_{\mathbb{V}_\rho^{(k)}} \mathcal{R}_{10}^{(kN)}(u, \tau) \dots \mathcal{R}_{L0}^{(kN)}(u, \tau), \quad (3.21)$$

where

$$\mathcal{R}_{j0}^{(kN)}(u, \tau) = \tau^{-H_0} P_{j0} \mathbb{R}_{j0}^{(N-k)}(u - \sigma_{N-k} + \rho_{N-k}) \dots \mathbb{R}_{j0}^{(N)}(u - \sigma_N + \rho_N).$$

We remind that  $\sigma = (\sigma_1, \dots, \sigma_N)$  are the parameters of the representation in the quantum space.  $\mathbb{R}_{j0}^{(k)}(u)$  are the factorizing operators (3.3) ¶. The proof of (3.20) follows exactly the proof of Theorem 2 in Ref. [14] where details can be found.

- The operators  $W_{\rho_k}^{(k)}(u)$ ,  $k = 1, \dots, N-1$  are finite in the limit  $\tau \rightarrow 1$ .

The finiteness of  $W_{\rho_k}^{(k)}(u, \tau = 1)$  follows from the fact that the trace in (3.20) involves only finite sums (see Ref. [14], Theorem 2) and therefore the  $\tau$ -regulator can be safely removed.

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<sup>‡</sup> We remind that we consider a homogeneous spin chain. This restriction is in no way a principal one but allows one to keep some expressions shorter.

<sup>§</sup>In particular, the  $SL(N, \mathbb{C})$  induced normalization introduced in [14] gives rise to Eq. (3.19).

¶ The operator  $\mathcal{R}_{j0}^{(kN)}(u, \tau)$  is defined on the tensor product of two generic spaces  $\mathbb{V} \otimes \mathbb{V}$  so that the permutation operator is unambiguously defined, however it has invariant subspace  $\mathbb{V} \otimes \mathbb{V}_\rho^{(k)}$



- The transfer matrix of  $k$ -th level (3.20) admits the following (alternating sum) representation in terms of the generic transfer matrices

$$\mathsf{T}_{\boldsymbol{\rho}}^{(k)}(u, \tau) = \sum_{P \in P_{k+1}} (-1)^{\text{sign}(P)} \mathsf{T}_{P\boldsymbol{\rho}}(u, \tau), \quad (3.22)$$

where the sum runs over all permutations of the set  $(\rho_{N-k}, \dots, \rho_N)$ , i.e.

$$P\boldsymbol{\rho} = (\rho_1, \dots, \rho_{N-k-1}, \rho'_{N-k}, \dots, \rho'_N)$$

and  $\text{sign}(P)$  is the parity of the permutation. Making use of the factorized representation for the generic transfer matrix (3.9) one derives from Eq. (3.22)

$$W_{\boldsymbol{\rho}_k}^{(k)}(u, \tau) = (\mathcal{P}\tau^H)^{-k} \det \begin{vmatrix} \mathcal{Q}_{N-k}(u + \rho_{N-k}, \tau) & \dots & \mathcal{Q}_{N-k}(u + \rho_N, \tau) \\ \vdots & \ddots & \vdots \\ \mathcal{Q}_N(u + \rho_{N-k}, \tau) & \dots & \mathcal{Q}_N(u + \rho_N, \tau) \end{vmatrix}. \quad (3.23)$$

We remind that the parameters  $\boldsymbol{\rho}$  in the above expression satisfy the condition  $\rho_{mm+1} = n_m \in \mathbb{N}$ ,  $m \geq N - k$ .

Equation (3.22) follows from the Bernstein - Gelfand - Gelfand resolution for the finite dimensional modules [17].

## 4 BGG resolution

We give here some details of Bernstein - Gelfand - Gelfand construction for the  $sl(N)$  algebra. Let  $v_{\boldsymbol{\rho}}$  be a finite dimensional module and  $\mathbb{V}_{\boldsymbol{\rho}}$  is a generic  $sl(N)$  module (2.1). Any permutation  $P$  can be represented as a composition of some number of permutations  $P_{kk+1}$ ,  $k = 1, \dots, N-1$ . The minimal number of such permutation is called the length of the permutation  $P$ ,  $\ell(P)$ .

Let  $\mathbb{V}_k$  be the direct sum of the spaces  $\mathbb{V}_{P\boldsymbol{\rho}}$ , where  $\ell(P) = k$

$$\mathbb{V}_k = \sum_{P, \ell(P)=k} \oplus \mathbb{V}_{P\boldsymbol{\rho}}. \quad (4.1)$$

That is  $\mathbb{V}_0 = \mathbb{V}_{\boldsymbol{\rho}}$ ,  $\mathbb{V}_1 = \mathbb{V}_{P_{12}\boldsymbol{\rho}} \oplus \mathbb{V}_{P_{23}\boldsymbol{\rho}} \oplus \dots \oplus \mathbb{V}_{P_{N-1N}\boldsymbol{\rho}}$  and so on. The last space has index  $s = N(N-1)/2$ ,  $\mathbb{V}_s = \mathbb{V}_{\rho_N \rho_{N-1} \dots \rho_1}$ .

The exact sequence constructed in Ref. [17] has the following form

$$0 \longrightarrow v_{\boldsymbol{\rho}} \xrightarrow{\varepsilon} \mathbb{V}_0 \xrightarrow{d_1} \mathbb{V}_1 \xrightarrow{d_2} \dots \xrightarrow{d_s} \mathbb{V}_s \longrightarrow 0 \quad (4.2)$$

Here the mapping  $\varepsilon$  is the natural embedding of the module  $v_{\boldsymbol{\rho}}$  into  $\mathbb{V}_0$ . The operator  $d_k$  is the  $n_k \times n_{k-1}$  matrix, where  $n_k$  is the number of the permutation of the length  $k$  (the number of the spaces in the direct sum (4.1)). Let  $\boldsymbol{\rho}^i$  and  $\boldsymbol{\rho}^j$  be the parameters of the generic modules  $[\mathbb{V}_k]_i$  and  $[\mathbb{V}_{k-1}]_j$ , respectively, (i.e.  $[\mathbb{V}_k]_i = \mathbb{V}_{\boldsymbol{\rho}^i}$ ,  $[\mathbb{V}_{k-1}]_j = \mathbb{V}_{\boldsymbol{\rho}^j}$ ). The entries  $[d_k]_{ij}$  are nonzero only if  $\boldsymbol{\rho}^i$  and  $\boldsymbol{\rho}^j$  differ by permutation of two elements,  $\boldsymbol{\rho}^i = P_{kn}\boldsymbol{\rho}^j$  and  $(\rho^j)_k - (\rho^j)_n = m > 0$  ( $k < n$ ). In this case  $[d_k]_{ij} = a_{ij}^k U_{kn}(\boldsymbol{\rho}^j)$ , where  $U_{kn}$  is the intertwining operator (2.8) and the coefficients  $a_{ij}^k$  are plus or minus one. For instance, the first operator can be chosen as  $[d_1]_{i1} = D_i^{\rho_{i+1}}$ ,  $i = 1, \dots, N-1$ .

Let us now fix the indices  $i$  and  $m$  and consider the matrix elements  $[d_k]_{ij}$  and  $[d_{k-1}]_{jm}$ . It was shown in Ref. [17] that either all these elements are zero for all  $j$  or there exist exactly two numbers  $j_1$  and  $j_2$  for which  $[d_k]_{ij}$  and  $[d_k]_{jm}$  are nonzero. Moreover, the numbers  $a_{ij}^k = \pm 1$  can be chosen in such a way that the product  $a_{ij_1}^k a_{ij_2}^k a_{j_1 m}^{k-1} a_{j_2 m}^{k-1} = -1$ . This property guarantees that  $[d_k]_{ij_1} [d_{k-1}]_{j_1 m} + [d_k]_{ij_2} [d_{k-1}]_{j_2 m} = 0$  and hence  $d_k d_{k-1} = 0$ , i.e.  $\text{Im } d_{k-1} \subset \ker d_k$ .

The sequence (4.2) is called a (BGG) resolution of the finite - dimensional module. It was proved by Bernstein, Gelfand and Gelfand that it is exact sequence, i.e.  $\ker d_k = \text{Im } d_{k-1}$  for all  $k$ .

Let  $\mathbf{T}_k(u, \tau)$  be a monodromy matrix on the  $\mathbb{V}_q \otimes \mathbb{V}_k$ , where  $\mathbb{V}_q$  is the quantum space of the spin chain model and  $\mathbb{V}_k$  is the space (4.1). The monodromy matrix  $\mathbf{T}_k(u)$ <sup>||</sup> is a diagonal  $n_k \times n_k$  matrix. Its entries are ordinary monodromy matrices,  $[\mathbf{T}_k(u)]_{ii} = \mathbf{T}_k^{(i)}(u)$ , which are defined on the tensor product  $\mathbb{V}_q \otimes [\mathbb{V}_k]_i$ . Obviously, the trace of monodromy matrix  $\mathbf{T}_k(u)$  over the space  $\mathbb{V}_k$  is given by the sum of transfer matrices  $\mathbf{T}_{P_i \rho}(u)$  (see Eq. (3.8))

$$\text{tr}_{\mathbb{V}_k} \mathbf{T}_k(u) = \sum_{i=1}^{n_k} \mathbf{T}_{P_i \rho}(u). \quad (4.3)$$

Next, since  $\ker d_k$  is the invariant subspace of  $\mathbb{V}_{k-1}$  the trace decays into the traces over  $\ker d_k$  and the factor space  $\tilde{\mathbb{V}}_{k-1} = \mathbb{V}_{k-1} / \ker d_k = \text{Im } d_k$ . So long as  $d_k \mathbf{T}_{k-1} = \mathbf{T}_k d_k$  one obtains

$$\text{tr}_{\tilde{\mathbb{V}}_{k-1}} \mathbf{T}_{k-1}(u) = \text{tr}_{\text{Im } d_k} \mathbf{T}_k(u) = \text{tr}_{\ker d_{k+1}} \mathbf{T}_k(u)$$

and hence

$$\text{tr}_{\mathbb{V}_{k-1}} \mathbf{T}_{k-1}(u) = \text{tr}_{\ker d_k} \mathbf{T}_{k-1}(u) + \text{tr}_{\ker d_{k+1}} \mathbf{T}_k(u). \quad (4.4)$$

Taking into account the boundary conditions

$$t_\rho(u) = \text{tr}_{\ker d_1} \mathbf{T}_0(u), \quad \text{tr}_{\ker d_{s+1}} \mathbf{T}_s(u) = \text{tr}_{\mathbb{V}_s} \mathbf{T}_s(u)$$

one derives from Eqs. (4.3), (4.4)

$$t_\rho(u) = \mathbf{T}_\rho^{(N-1)} = \sum_{k=0}^s (-1)^k \mathbf{T}_k(u) = \sum_P (-1)^{\text{sign}(P)} \mathbf{T}_{P \rho}(u), \quad (4.5)$$

where the last sum runs over all permutations.

Let us notice that the derivation of the representation (3.22) for the higher level transfer matrices,  $\mathbf{T}_\rho^{(k)}(u)$ ,  $k < N - 1$  follows exactly the same lines. It is sufficient to notice that the submodule  $\mathbb{V}_\rho^{(k)}$  is given by the tensor product (2.13) where the space  $v_k$  is a finite dimensional  $sl(k+1)$  module.

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<sup>||</sup>For brevity, till the end of this section we will omit the  $\tau$ -dependence.

## 5 Baxter Equation and Nested Bethe Ansatz

The finite dimensional module  $v_\rho$ ,  $\rho_{kk+1} = n_k = m_k + 1 \in \mathbb{N}$  corresponds to the highest weight  $\chi = (m_1, \dots, m_{N-1})$  or to the Young tableau specified by the partitions

$$\ell = \{\ell_1, \ell_2, \dots, \ell_{N-1}\},$$

where  $\ell_k = \sum_{i=k}^{N-1} m_i$  is the length of the  $k$ -th row in the tableau. The finite dimensional transfer matrix corresponding to the Young tableau  $\ell$  has the form

$$t_\ell(u + f_\ell, \tau) = (\mathcal{P}\tau^H)^{-N+1} \det \begin{vmatrix} \mathcal{Q}_1(u + l_1, \tau) & \mathcal{Q}_1(u + l_2, \tau) & \dots & \mathcal{Q}_1(u, \tau) \\ \mathcal{Q}_2(u + l_1, \tau) & \mathcal{Q}_2(u + l_2, \tau) & \dots & \mathcal{Q}_2(u, \tau) \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{Q}_N(u + l_1, \tau) & \mathcal{Q}_N(u + l_2, \tau) & \dots & \mathcal{Q}_N(u, \tau) \end{vmatrix}, \quad (5.1)$$

where  $l_k = \rho_k - \rho_N = \ell_k - k + N$  and

$$f_\ell \equiv -\rho_N = \frac{1}{N} \sum_{k=1}^{N-1} \ell_k = \frac{1}{N} (m_1 + 2m_2 + \dots + (N-1)m_{N-1}).$$

The transfer matrix corresponding to the *null* Young tableau,  $\ell_k = 0$ ,  $k = 1, \dots, N-1$  (i.e. to the trivial one dimensional representation),  $t_{\ell=0}(u, \tau)$ , is proportional to the unit operator on the quantum space,

$$t_{\ell=0}(u, \tau) = \Delta(u, \tau) \mathbb{1}. \quad (5.2)$$

Thus, it follows from (5.1) that the determinant of a  $N \times N$  matrix  $A(u, \tau)$ ,  $A_{ij}(u, \tau) = \mathcal{Q}_i(u + N - j, \tau)$ , is given by

$$\det ||A(u, \tau)|| = \Delta(u, \tau) (\mathcal{P}\tau^H)^{N-1}. \quad (5.3)$$

The equation (5.3) is usually referred to as the Wronskian relation.

The determinant representation (5.1) gives rise to the self-consistency equations (Baxter equations) involving the Baxter  $\mathcal{Q}$ -operators and the finite dimensional transfer matrices. Indeed, let  $t_k(u)$  be the transfer matrix corresponding to the Young tableau with one column and  $k$  boxes,  $0 \leq k \leq N$ , (the transfer matrices with  $k = 0$  or  $k = N$  correspond to the trivial representation, i.e.  $t_0(u, \tau) = t_N(u, \tau) = \Delta(u, \tau)$ .) Let  $B_j(u)$  be a  $(N+1) \times (N+1)$  matrix

$$(B_j)_{ik}(u, \tau) = \begin{cases} \mathcal{Q}_i(u + N + 1 - k, \tau), & i \leq N, \\ \mathcal{Q}_j(u + N + 1 - k, \tau), & i = N + 1 \end{cases}. \quad (5.4)$$

By construction the matrix  $B_j$  has two identical lines, hence  $\det B_j(u, \tau) = 0$ . Expansion of  $\det B_j(u, \tau)$  over the elements of the last line gives rise to the following relation

$$\sum_{k=0}^N (-1)^k t_k(u + k/N, \tau) \mathcal{Q}_j(u + N - k, \tau) = 0, \quad (5.5)$$

which is the  $N$ -th order difference (Baxter) equation on the operator  $\mathcal{Q}_j(u, \tau)$ . Let us notice that except the  $sl(2)$  case the Baxter operators  $\mathcal{Q}_k(u, \tau)$  with  $k < N$  are singular in the limit  $\tau \rightarrow 1$ . However, as follows from Eqs. (3.23), (5.1) their antisymmetrized products are free from singularities.

To establish connection with the Nested Bethe Ansatz approach let us consider the minors of the matrix corresponding to the null Young tableau. Namely, we define new set of the operators  $\hat{\mathcal{Q}}_k(u, \tau)$ ,  $k = 2, \dots, N$  by

$$\hat{\mathcal{Q}}_k(u, \tau) = \det \begin{vmatrix} \mathcal{Q}_k(u + N - 1, \tau) & \dots & \mathcal{Q}_k(u + k - 1, \tau) \\ \vdots & \ddots & \vdots \\ \mathcal{Q}_N(u + N - 1, \tau) & \dots & \mathcal{Q}_N(u + k - 1, \tau) \end{vmatrix}, \quad (5.6)$$

that is

$$\hat{\mathcal{Q}}_N(u, \tau) = \mathcal{Q}_N(u + N - 1, \tau),$$

$$\hat{\mathcal{Q}}_{N-1}(u, \tau) = \mathcal{Q}_{N-1}(u + N - 1, \tau) \mathcal{Q}_N(u + N - 2, \tau) - \mathcal{Q}_N(u + N - 1, \tau) \mathcal{Q}_{N-1}(u + N - 2, \tau)$$

and so on. The operators  $\hat{\mathcal{Q}}_k$  arise as the factorizing operators for the  $k$ -th level transfer matrices, such that  $\rho_{mm+1} = 1$ , for  $m \geq N - k$ ,

$$\mathsf{T}_{\rho}^{(k)}(u, \tau) \sim \mathcal{Q}_1(u + \rho_1, \tau) \dots \mathcal{Q}_{N-k-1}(u + \rho_{N-k-1}, \tau) \hat{\mathcal{Q}}_{N-k}(u + \delta_k, \tau), \quad (5.7)$$

where  $\delta_k = \rho_N + k + 1 - N$ . Hence, as it was explained in Sec. 3.1 these operators are finite in the limit  $\tau \rightarrow 1$ .

Further, with the help of the identity (A.2) one can express the ratio of transfer matrices  $t_1(u + 1/N)/t_0(u) \equiv \bar{t}_1(u)$  as follows (henceforth we will omit  $\tau$ -dependence),

$$\bar{t}_1(u) = \frac{\Delta(u+1)}{\Delta(u)} \frac{\hat{\mathcal{Q}}_2(u-1)}{\hat{\mathcal{Q}}_2(u)} + \sum_{k=2}^{N-1} \frac{\hat{\mathcal{Q}}_k(u+1)}{\hat{\mathcal{Q}}_k(u)} \frac{\hat{\mathcal{Q}}_{k+1}(u-1)}{\hat{\mathcal{Q}}_{k+1}(u)} + \frac{\hat{\mathcal{Q}}_N(u+1)}{\hat{\mathcal{Q}}_N(u)}, \quad (5.8)$$

where  $\Delta(u) = t_0(u)$ . Introducing the notations

$$\Lambda_k(u) = \frac{\hat{\mathcal{Q}}_k(u+1)}{\hat{\mathcal{Q}}_k(u)} \frac{\hat{\mathcal{Q}}_{k+1}(u-1)}{\hat{\mathcal{Q}}_{k+1}(u)}, \quad k = 1, \dots, N, \quad (5.9)$$

where  $\hat{\mathcal{Q}}_1 = \Delta(u)$  and  $\hat{\mathcal{Q}}_{N+1}(u) = 1$  one can rewrite (5.8) in the form

$$\bar{t}_1(u) = \Lambda_1(u) + \Lambda_1(u) + \dots + \Lambda_N(u), \quad (5.10)$$

which is the well-known expression for the transfer matrix in the Nested Bethe Ansatz approach [48, 49].

Making use of the identity (A.2) and the determinant representation (5.1) one can express the transfer matrices  $\bar{t}_k(u) = t_k(u + k/N)/\Delta(u)$  corresponding to the Young tableau with one column and  $k$  boxes in terms of ratios (5.9)

$$\bar{t}_k(u) = \sum_{m_1=1}^N \sum_{m_2=m_1+1}^N \dots \sum_{m_k=m_{k-1}+1}^N \Lambda_{m_1}(u) \Lambda_{m_2}(u-1) \dots \Lambda_{m_k}(u-k+1). \quad (5.11)$$

The representation for an arbitrary finite dimensional transfer matrix in terms of  $\Lambda_k$  were obtained by Kirillov and Reshetikhin, see Ref. [12, 13] for details.

The finite dimensional transfer matrices satisfy an infinite set of the functional (fusion) relations [5, 7, 9, 11]. The representation (5.1) allows one to generate such relations in a straightforward way with the help of the following determinant identity

$$\begin{aligned} \|a_1, a_2, a_3, \dots, a_N\| \|b_1, b_2, a_3, \dots, a_N\| &= \|a_1, b_1, a_3, \dots, a_N\| \|a_2, b_2, a_3, \dots, a_N\| \\ &+ \|a_1, b_2, a_3, \dots, a_N\| \|b_1, a_2, a_3, \dots, a_N\|. \end{aligned} \quad (5.12)$$

Here  $\|a_1, a_2, a_3, \dots, a_N\|$  stands for the determinant of  $N \times N$  matrix with the columns  $a_1, \dots, a_N$ , and similar for others. For example, let us assume that the Young tableau  $\ell$  has more than one column and that the last row with more than one box in the row has index  $p$ , i.e.  $\ell_p > 1$  and  $\ell_{p+1} \leq 1$ . Using Eq. (5.12) one can derive the following quadratic relation

$$\tilde{t}_\ell(u) \tilde{t}_{\ell'}(u+1) = \tilde{t}_{\ell'}(u+1) \tilde{t}_{\ell_1}(u) - \tilde{t}_{\ell_-}(u) \tilde{t}_{\ell_+}(u+\delta), \quad (5.13)$$

where we put  $\tilde{t}_\ell(u) = t_\ell(u + f_\ell)$ . The Young tableau  $\ell_1$  is obtained from the tableau  $\ell$  by deleting one box from the  $p$ -th row. The primed Young tableau  $\ell'(\ell'_1)$  is obtained from the corresponding unprimed tableau  $\ell(\ell_1)$  by crossing out the first column. The Young tableau  $\ell_-$  is obtained from the tableau  $\ell$  by crossing out all rows except first  $p-1$ , i.e.  $\ell_- = \{\ell_1, \dots, \ell_{p-1}, 0, \dots, 0\}$ . To construct the table  $\ell_+$  one has to cross out all columns containing  $N$ -boxes from the auxiliary table with  $N$  rows

$$L = \{\ell_1 - 1, \dots, \ell_{p-1} - 1, \ell_p - 1, \ell_p - 1, \ell_{p+1}, \dots, \ell_{N-1}\}.$$

The parameter  $\delta = 1 + L_N$ , i.e.  $\delta = 1$  for  $\ell_{N-1} = 0$ ,  $\delta = 2$  for  $\ell_{N-1} = 1$  and  $\delta = \ell_p$  if  $p = N - 1$ . Let us introduce ordering for Young tableaux as follows:  $\ell' < \ell$  if the first nonzero entry in  $\ell - \ell' = \{\ell_1 - \ell'_1, \ell_2 - \ell'_2, \dots\}$  is positive. It is easy to see that the Young tableau  $\ell$  is the maximal one among the Young tableaux appearing in Eq. (5.13). It means that the arbitrary transfer matrix corresponding to the Young tableau with more than one column can be expressed in terms of the "one column" transfer matrices. Such expression was obtained by Bazhanov and Reshetikhin [13]

$$\bar{t}_\ell(u) = \det A(u), \quad A_{ij}(u) = \bar{t}_{m_j - j + i}(u + i - 1), \quad 1 \leq i, j \leq \ell_1. \quad (5.14)$$

Here  $\bar{t}_\ell(u) \equiv t_\ell(u + f_\ell)/t_0(u)$ ,  $m_j$  is the length of the  $j$ -th column in the Young tableau  $\ell$ . The "one-column" transfer matrices  $\bar{t}_m(u)$  are defined in (5.11) for  $0 \leq m \leq N$  and  $\bar{t}_m(u) \equiv 0$  for  $m < 0$  and  $m > N$ . To verify (5.14) it is sufficient to show that it solves Eq. (5.13) that can be done with the help of the identity (5.12).

## 6 Summary

We study the properties of transfer matrices for the  $sl(N)$  spin chain models. It was shown in Ref. [14] that the transfer matrices with a generic (infinite-dimensional) auxiliary space are

factorized into the product of  $N$ -commuting Baxter  $\mathcal{Q}$ -operators. Both transfer matrices and Baxter operators depend on the regularization parameter  $\tau$  which ensures the convergence of the corresponding traces. The regularized transfer matrices are invariant only with respect to the Cartan generators of the  $sl(N)$  algebra and become singular when the regularization is removed,  $\tau \rightarrow 1$ . The same concerns the Baxter operators  $\mathcal{Q}_j(u, \tau)$  which are singular at  $\tau \rightarrow 1$ , except the operator  $\mathcal{Q}_N(u, \tau)$  which is finite in this limit. To find operators which survive removing of the regularization we considered the transfer matrices with the special auxiliary spaces. It was shown that the transfer matrix with the auxiliary space  $\mathbb{V}_{\boldsymbol{\rho}}^{(k)}$  (see Eq. (2.11)) is factorized into the product of  $N - k$  commuting operators:  $\mathcal{Q}_j(u, \tau)$ ,  $j = 1, \dots, N - k - 1$  and one new operator which is finite in the limit  $\tau \rightarrow 1$ . Moreover, new operator can be represented as the alternating sum over the product of the Baxter operators  $\mathcal{Q}_j(u, \tau)$ ,  $j = N - k, \dots, N$ . This representation follows from the BGG-resolution for the finite dimensional  $sl(N)$  modules and the factorized representation for the generic transfer matrices.

We defined new set of the commuting operators  $\hat{\mathcal{Q}}_j(u, \tau)$ ,  $j = 2, \dots, N$  which are finite in the limit  $\tau \rightarrow 1$ . The finite dimensional transfer matrices can be expressed as the sum over ratios of  $\hat{\mathcal{Q}}_j(u, \tau)$  operators. The corresponding expressions have the same form as in the Nested Bethe Ansatz approach.

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## 1 Appendix: Determinant identity

Let  $A_k(j_1, \dots, j_k)$  be a determinant of  $k \times k$  matrix

$$A_k(j_1, \dots, j_k) = \det \begin{vmatrix} a_1(j_1) & \dots & a_1(j_k) \\ \vdots & \ddots & \vdots \\ a_k(j_1) & \dots & a_k(j_k) \end{vmatrix}, \quad (\text{A.1})$$

and similarly  $A_{k-1}(j_2, \dots, j_k)$  is the determinant of  $(k - 1) \times (k - 1)$  matrix  $a_{ik} = a_i(j_k)$ ,  $2 \leq i, j \leq k$ . Then the following identity holds

$$\begin{aligned} \frac{A_k(j_0, j_2, \dots, j_k)}{A_k(j_1, j_2, \dots, j_k)} &= \frac{A_{k-1}(j_2, \dots, j_k)}{A_{k-1}(j_1, \dots, j_{k-1})} \\ &+ \frac{A_{k-1}(j_0, j_2, \dots, j_{k-1})}{A_{k-1}(j_1, \dots, j_{k-1})} \frac{A_k(j_0, j_1, \dots, j_{k-1})}{A_k(j_1, j_2, \dots, j_k)}. \end{aligned} \quad (\text{A.2})$$

It easily follows from the identity (5.12).

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